Methods of scaling Whistlers in the Absence of the Initiating Sferic and Nose Frequency

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Four methods of scaling whistler sonagrams in the absence of initiating sferic and nose are compared. It is concluded that, while all give acceptable results, the method of *Ho & Bernard* (1973) is most economical of computer time and scaling time.

Vier metodes om sonagramme van fluiters in afwesigheid van inisiërende sferieksteuring en neusfrekwensie te skaleer, word vergelyk. Alhoewel met al die metodes aanvaarbare resultate verkry word, blyk die metode van Ho & Bernard *(1973) wat rekenaartyd en skaleertyd betref die voordeligste te wees.*

Introduction

Whistler sonagrams are commonly used in order to obtain information about the electron density in the plasmasphere *(Helliwell*, 1965). If a suitable model is assumed, a knowledge of the nose frequency and position of the initiating sferic for each whistler trace allows the L value of the duct in which the whistler has propagated, and the electron density distribution in the duct to be deduced *(Park*, 1972). Frequently both initiating sferic and nose are absent on the trace and recently there has been great interest in curve-fitting techniques for deducing the position of the sferic and the nose frequency *(Rycroft & Mathur*, 1973; *Ho & Bernard*, 1973). These are based on one of two fitting formulae *(Dowden & Allcock*, 1971; *Bernard*, 1973).

In this paper four methods of deducing the nose frequency and sferic position are discussed and compared. The methods are tested on synthetic data computed from a model plasmasphere. All methods give similar results and the choice of method is a matter of convenience. It is concluded that the most economical of computer time and manpower is that of *Ho & Bernard* (1973) with a modification which makes more efficient use of the data available. This method is now in use for scaling data from Sanae, Antarctica.

The whistler group delay and plasmasphere models

Whistlers which have travelled in ducts in the plasmasphere have a time delay which is frequency dependent. The expression for the delay, τ, is *(Helliwell*, 1965, p. 182)

$$\tau = 2c\, f^{-1/2} \int_{\text{path}} \frac{f_N f_H}{(f_H - f)^{3/2}}\, ds \qquad (1)$$

where f is the wave frequency, f_N the plasma frequency, f_H the electron gyrofrequency and c the free space speed of light. The integral is taken along the path, with ds the element of path length. The quantities f_N and f_H are functions of s. The dispersion $D = \tau\sqrt{f}$, is a parameter frequently used.

In standard methods of data reduction approximations are made to this law and whistler traces are analysed on the basis of assumed models. It is normally assumed that the earth's magnetic field is that of a centred dipole and that the plasmasphere is in ambipolar diffusive equilibrium, with ions and electrons constrained to move along the magnetic field lines *(Angerami & Thomas*, 1964).

In this paper synthetic whistlers have been computed from equation (1) by evaluating the integral using Simpson's rule. These have then been scaled by different techniques, and the results compared. The model used was that of *Rycroft & Alexander* (1969) for winter night. It is a diffusive equilibrium model which is representative of average conditions in the plasmasphere. It is assumed that a duct exists at intervals of 0,5 in L between L = 2,5 and L = 6. Some of the resulting synthetic whistlers are shown in Fig. 1 and may be compared with a typical sonagram from Sanae shown in Fig. 2.

Fitting formulae for the whistler group delay

The formula of Dowden & Allcock

On empirical grounds *Dowden & Allcock* (1971) have proposed that Q, the reciprocal of the dispersion, is linear in f. This leads to an expression for delay of the form

$$\tau = \tau_0 + \frac{1}{\sqrt{f}}\frac{D_0}{1 - f/af_n} \qquad (2)$$

or

$$Q\ (\equiv (\tau - \tau_0)\sqrt{f}) = \frac{1}{D_0}(1 - f/af_n) \qquad (3)$$

Here D_O is the zero order dispersion *(Storey* 1953), τ_O the position of the initiating sferic and f_n the nose frequency or frequency of minimum delay. This occurs where $d\tau/df = 0$ which condition gives

$$a = 3.$$

On statistical grounds, by scaling real data, Dowden and Allcock obtained a value $a = 3{,}1 \pm 0{,}04$. The slightly higher value for a can be regarded as a correction for non-linearity of Q(f), averaged for the conditions which they were considering.

It is of interest to investigate the theoretical reason for the excellent fit to experimental data of equation (2). *Storey* (1957) has expanded the integral in equation (1) in powers of f to give for the dispersion an expression of the form

$$D = (2c)^{-1} \int (f_N/f_H^{\frac{1}{2}}) \{1 - (3/2)(f/f_H) + (15/8)(f/f_H)^2 + \ldots\} \, ds$$

$$= (2c)^{-1} \{ \int (f_N/f_H^{\frac{1}{2}}) \, ds - (3f/2) \int (f_N/f_H^{\frac{3}{2}} ds) + (15f^2/8) \int (f_N/f_H^{\frac{5}{2}}) \, ds + \ldots\}$$

$$= (2c)^{-1} \{I_o - (3/2) I_1 (f/f_{HE}) + (15/8) I_2 (f/f_{HE})^2 + \ldots\} \quad (4)$$

where f_{HE} is the gyrofrequency on the path at the equator, and I_O, I_1, and I_2 are given in the appendix.

For $f \ll f_n$ this approximation is very good but for values of $f \sim f_n$ many terms are needed. The expression (3), however, is an excellent approximation over a wide range of frequencies. The reason for this can be seen if we regard $f_N/f_H{}^{1/2}$ as a weighting function and express I_1 and I_2 in terms of I_O and weighted values of $1/f_H$ and $1/f_H{}^2$:

$$I_1 = \left\langle \frac{1}{f_H} \right\rangle I_o$$

$$I_2 = \left\langle \frac{1}{f_H^2} \right\rangle I_o.$$

If equation (4) is inverted we get

$$Q = \frac{2c}{I_o} \left\{ 1 + \frac{3}{2} \frac{I_1}{I_o} \left(\frac{f}{f_{HE}}\right) + \frac{18 I_1^2 - 15 I_o I_2}{8 I_o^2} \left(\frac{f}{f_{HE}}\right)^2 + \ldots \right\}$$

$$= \frac{2c}{I_o} \left\{ 1 + \frac{3}{2} \left\langle \frac{1}{f_H} \right\rangle \frac{f}{f_{HE}} + \frac{3}{8} \left[6 \left\langle \frac{1}{f_H} \right\rangle^2 - 5 \left\langle \frac{1}{f_H^2} \right\rangle \right] \left(\frac{f}{f_{HE}}\right)^2 + \ldots \right\} \quad (5)$$

The quantities $\left\langle \frac{1}{f_H} \right\rangle^2$ and $\left\langle \frac{1}{f_H^2} \right\rangle$

are positive and do not differ very much in magnitude for all reasonable distributions so that the coefficient of the term in $(f/f_{HE})^2$ is very small in equation (5). This is not the case in equation (4). In the appendix some computations of the size of the coefficients

$$\frac{18 I_1^2 - 15 I_o I_2}{8 I_o^2} \text{ and } \frac{15 I_2}{8 I_o}$$

are presented for a typical model. These show it to be very small compared with unity and with $3I_1/2I_O$.

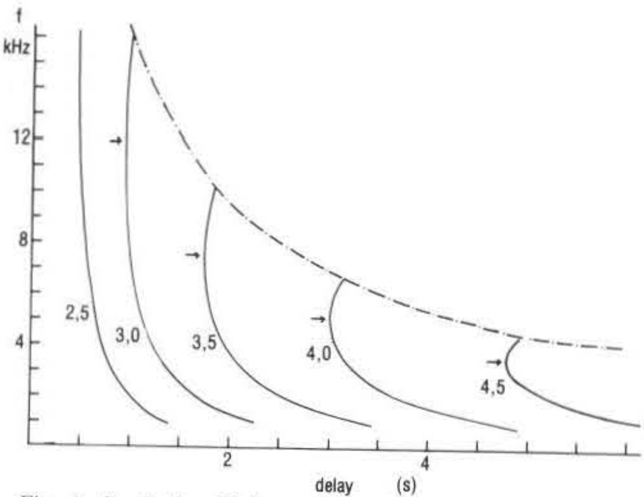

Fig. 1. Synthetic whistlers computed in a model ionosphere (Rycroft and Alexander model 4). The parameter beside each curve is the value of L.

We thus see that there is a good reason for the fact that the Dowden and Allcock formula is a good approximation to the whistler dispersion law.

Bernard's formula

Bernard (1973) has suggested a formula of the form

$$D \equiv D_o \frac{f_{HE} - Af}{f_{HE} - f} \quad (6)$$

where D_O is the zero order dispersion and

$$f_{Ho} = f_n/\Lambda_n, \; A = \frac{3\Lambda_n - 1}{\Lambda_n (1 + \Lambda_n)}, \; \Lambda_n = f_n/f_{HE}.$$

This he justifies theoretically and it provides a good fit over a wide range of parameters.

Techniques of whistler analysis

In all the techniques of this section it is assumed that initiating sferic and nose are both absent.

Method of Ho & Bernard (1973) (Method 1)

Here Bernard's fitting formula (6) is used. A value is assumed for Λ_n and A. The time delay referred to an

Table 1

L	$3I_1/2I_O$	$15 I_2/8 I_O$	$\frac{18 I_2 - 15 I_0 I_2}{8 I^2}$
1,5	1,080	0,996	0,171
2,0	0,874	0,746	0,017
2,5	0,798	0,677	−0,040
3,0	0,767	0,654	−0,065
3,5	0,755	0,648	−0,077
4,0	0,752	0,649	−0,083
4,5	0,753	0,652	−0,085
5,0	0,756	0,657	−0,085
5,5	0,760	0,662	−0,085
6,0	0,764	0,667	−0,084
6,5	0,768	0,673	−0,082
7,0	0,772	0,676	−0,080
7,5	0,777	0,682	−0,079
8,0	0,780	0,686	−0,077
8,5	0,784	0,690	−0,075
9,0	0,787	0,694	−0,074
9,5	0,790	0,697	−0,072
10,0	0,793	0,700	−0,071

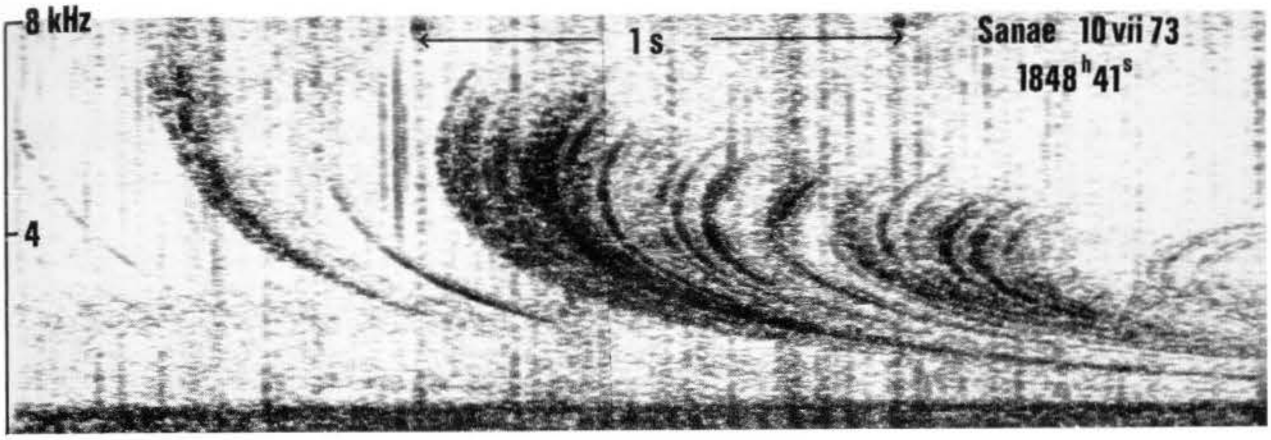

Fig. 2. Typical whistler sonagram from Sanae.

arbitrary origin is then

$$\tau = \tau_0 + \frac{D_0}{\sqrt{f}\,\{1 - f/3{,}1\, f_n\}}$$

Here τ_O is the unknown sferic position. Thus there are three unknown parameters, D_O, f_{HE}, and τ_O. Method 1 requires the measurement of three experimental pairs of values, (τ, f). The three equations are solved to give the three unknown parameters and hence τ_O and f_n.

Method of Rycroft & Mathur (1973) (Method 2)

In this method fitting formula (2) is used. A position is assumed for the sferic and from a set of n data points (τ_i, f_i). With the assumption that τ_O is zero, a linear least squares fit to equation (3) is made. The process is repeated for various sferic positions and that value of τ_O which minimizes the standard error of the gradient selected.

Method 3

This previously unpublished method uses equation (2) directly. This involves a non-linear least squares fit. The quantity to be minimized is

$$\sum_{i=1}^{n} \left\{\tau_i - \tau_0 - \frac{1}{\sqrt{f_i}}\,\frac{D_0}{1 - f_i/3{,}1\, f_n}\right\}^2$$

where there are n pairs of data points (τ_i, f_i). This can be regarded as a surface in the parameter space $\{\tau_O, D_O, f_N\}$. Standard techniques for a numerical search for the minima of this function exist *(Bevington*, 1969). These are used to give values for the parameters that fit best and for an estimate of the error in each parameter. For a multiple-path whistler, where there are many traces arising from a single lightning flash, a significant improvement can be made by taking note of the fact that the sferic position is the same for each trace. The value of τ_O is obtained independently for each trace together with its standard deviation. A weighted mean value is then found for τ_O and the record rescaled using a linear fit of the data to equation (3).

Method 4

This is an extended version of method 1. For each trace in a multi-path whistler a value of τ_O is obtained by method 1. Unlike method 3 there is no estimate of the standard deviation. It can, however, be assumed that the fractional error is the same in each case. This leads to the assumption that the absolute error in τ_O is proportional to the value of the delay at the nose. A mean value of τ_O is found with weights proportional to τ_N^{-2} and the data rescaled by *Bernard's* (1973) method. This is a more efficient use of the data than in the original method of *Ho & Bernard* (1973).

Results and conclusions

The methods described above have been applied to find the nose frequency and sferic position for each of the synthetic whistlers of Figure 1. Some results are shown in Table 2. It will be noted that all give consistent and satisfactory results here and this appears generally to be the case. Although in this particular example the sferic position obtained by method 4 does not appear as good as that obtained by other methods, the reverse is true in other examples, and there is no clear advantage for any of the methods on the grounds of accuracy alone. A decision on suitability can thus be made on grounds of convenience. The following general statements can be made:

(i) Method 1 (and hence method 4) requires relatively little scaling time. Only three points need be read from each trace as compared with 10 to 20 for methods 2 and 3.

(ii) Method 2 (and method 4) require substantially less computer time. Method 3 is very expensive on computer time.

(iii) Method 3 gives an estimate of the statistical error involved in the scaling and is the only one which does so. This error turns out to be much smaller than the estimated systematic errors due to the assumption of particular parameters for the model *(Park*, 1972). It thus does not seem worthwhile for routine analysis.

(iv) All methods appear to give consistent results when applied to real data from Sanae.

(v) Method 4, unlike method 1, uses the extra information available through the knowledge that the sferic position is the same for each trace of a multi-path whistler.

We conclude therefore, that the best method for routine analysis is method 4 (modified from *Ho & Bernard*, 1973). This method is now being applied in this laboratory for routine analysis of whistlers.

Table 2

Comparison of the nose frequency and sferic position computed by each of the methods in this paper. The actual sferic position is -0,5 s.

L	Actual	Nose frequency (kHz)				Sferic Position			
		Method				Method			
		1	2	3	4	1	2	3	4
2,5	20,5	17,2	15,7	15,5	17,2	−0,51	−0,55		
3,0	12,0	10,7	11,8	11,0	11,2	−0,54	−0,51		
3,5	7,6	7,6	7,4	7,4	7,4	−0,45	−0,56	−0,55	−0,37
4,0	5,1	5,1	4,9	5,0	5,1	−0,39	−0,65	±0,01	±0,06
4,5	3,6	3,6	3,4	3,5	3,6	−0,21	−0,86		
5,0	2,6	2,6	2,5	2,5	2,6	−0,10	−1,22		

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Appendix

If we adopt a typical diffusive equilibrium model for the plasma frequency, and dipole field variation for the gyrofrequency, then we may write

$$f_N = f_{NE}\, e^{2,5/L\cos^2\theta}$$

and

$$f_H = f_{HE}\,(4 - 3\cos^2\theta)^{1/2}/L^3\cos^6\theta$$

where f_{NE} and f_{HE} are the values of f_N and f_H at the magnetic equator on the field line defined by L.

Then, noting that

$$\frac{ds}{d\theta} = aL\,(4 - 3\cos^2\theta)^{1/2},$$

the quantities I_0, I_1, I_2 in equation 5 may be written

$$I_0 = f_{NE}\ a \int_0^{\Lambda'} \cos^4\theta\,(4 - 3\cos^2\theta)^{1/4} \exp(2,5/L\cos^2\theta)\, d\theta$$

$$I_1 = f_{NE}\ a \int_0^{\Lambda'} \cos^{10}\theta(4 - 3\cos^2\theta)^{-1/4} \exp(2,5/L\cos^2\theta)\, d\theta$$

$$I_2 = f_{NE}\ a \int_0^{\Lambda'} \cos^{16}\theta(4 - 3\cos^2\theta)^{-5/4} \exp(2,5/L\cos^2\theta)\, d\theta$$

where $\cos\Lambda' = 1{,}07/L^{1/2}$ and Λ' is the latitude where the field line reaches an altitude of 1 000 km. These have been evaluated by Simpson's rule and the quantities $3I_1/2I_0$, and $(18I_1^2 - 15\,I_0I_2)/8\,I_0^2$ are shown in Table 1. It will be observed that in the Storey expansion the coefficients of the first and second order terms are of order unity while for the Dowden and Allcock expansion the second order coefficient is about 0,1 of the first order coefficient. Of course this calculation omits the contribution of the ionosphere below 1 000 km to the dispersion. This is usually handled by using a simple correcting term such as that due to *Park* (1972).

A condition for the whistler to be ducted is that $f < \frac{1}{2} f_{HE}$ (*Smith*, 1961), thus the maximum size of the second order term in the Dowden and Allcock expansion is 0,25 of the coefficient i.e. a maximum error of only 2% is introduced over a very wide range of L values if this term is ignored. For the Storey expansion this maximum error is of the order of 20%. If we take the nose frequency as being typically $0{,}3f_{HE}$ we see that the error at the nose is 0,8%.

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